Generalized Virtual Polytopes and Quasitoric Manifolds

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Abstract—We develop a theory of volume polynomials of generalized virtual polytopes based on the study of topology of affine subspace arrangements in a real Euclidean space. We apply this theory to obtain a topological version of the Bernstein–Kushnirenko theorem as well as Stanley–Reisner and Pukhlikov–Khovanskii type descriptions for the cohomology rings of generalized quasitoric manifolds.

1. INTRODUCTION

In [24] A. V. Pukhlikov and the third author generalized the classical theory of finitely additive measures of convex polytopes and proposed a geometric construction for a virtual polytope as a Minkowski difference of two convex polytopes. Based on this notion, in [25] the same authors proved a Riemann–Roch type theorem linking integrals and integer sums of quasipolynomials over convex chains from a certain family. As a byproduct, they obtained a description of the cohomology ring of a complex nonsingular projective toric variety via a volume polynomial of a virtual polytope. A theory of mixed volumes of virtual convex bodies aimed at producing an "elementary" proof of the classical g-theorem was developed in [26]; this theory was motivated by the ideas of [25] and the approach of [22].

A topological generalization of a complex nonsingular projective toric variety is known in toric topology as a (quasi)toric manifold. It was introduced and studied along with its real counterpart, a small cover, in [7]. In particular, it was shown in [7] that the Stanley–Reisner description of the cohomology rings holds for quasitoric manifolds. Since that time, quasitoric manifolds and their generalization, torus manifolds [20, 10], have been intensively studied in toric topology and found numerous valuable applications in homotopy theory [6, 9, 8], unitary [5, 19] and special unitary bordism [18, 17], hyperbolic geometry [2–4], and other areas of research.

A remarkable property of torus manifolds is that they admit a combinatorial description similar to the one available for toric varieties. Namely, instead of a fan, it is based on the notions of a multi-fan and a multi-polytope, introduced and studied in [10]. A multi-fan is a collection of cones that can overlap each other, unlike the classical case of cones in an ordinary fan.

A multi-polytope is a multi-fan along with a collection of affine hyperplanes orthogonal to the linear spans of its rays. The relation between a multi-polytope and its multi-fan is similar to the one between a polytope and its normal fan. In [1] the theory of multi-polytopes was applied to prove a version of the Bernstein–Kushnirenko (or Bernstein–Khovanskii–Kushnirenko, BKK) theorem and the Pukhlikov–Khovanskii type description for the cohomology rings of quasitoric manifolds. On

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the other hand, a Stanley–Reisner type description for the cohomology of certain torus manifolds was obtained in [21] using methods and tools of the theory of manifolds with corners and equivariant topology.

Smooth structures on quasitoric manifolds were constructed in [5] by means of a topological analogue of the Cox construction, in which a coordinate subspace arrangement is replaced by a moment–angle manifold. By the result of [23], moment–angle complexes over star-shaped spheres have smooth structures. This allowed us in [15] to introduce the class of generalized quasitoric manifolds consisting of quotient spaces of moment–angle complexes over star-shaped spheres by freely acting compact tori of maximal possible rank. The class of generalized quasitoric manifolds is closely related to the class of topological toric manifolds introduced in [12]. Indeed, any generalized quasitoric manifold is a topological toric manifold with respect to the restriction of a smooth action of the group $(\mathbb{C}^*)^n$ to $(S^1)^n$. To emphasize this difference, here we keep using the term "generalized quasitoric manifold."

This paper is devoted to developing the theory of generalized virtual polytopes and applying it in order to obtain a topological version of the BKK theorem as well as Stanley–Reisner and Pukhlikov–Khovanskii type descriptions for the intersection rings of generalized quasitoric manifolds.

Generalized virtual polytopes and affine subspace arrangements. The first part of the paper is devoted to the theory of generalized virtual polytopes and integration of forms over them, based on studying the homotopy types of unions of affine subspace arrangements in real Euclidean spaces. The construction and the theory of generalized virtual polytopes were motivated by the properties of integral functionals on the space of smooth convex bodies. We discuss smooth convex bodies in Section 2.

Let Q be a polynomial of degree at most k (homogeneous polynomial of degree k) on $\mathbb{R}^n$, $\omega = dx_1 \wedge \ldots \wedge dx_n$ be the standard volume form on $\mathbb{R}^n$, and C_s be the cone of strictly convex bodies $\Delta \subset \mathbb{R}^n$ with smooth boundary. Then the function

$$F(\Delta) = \int_\Delta Q\omega$$

on the cone C_s is a polynomial of degree at most $k + n$ (homogeneous polynomial of degree $k + n$).

Now, to extend the domain of the integral functional to the entire vector space generated by the cone C_s, we introduce the notion of a *virtual convex body* as a formal difference of two convex bodies (with the usual identification $\Delta_1 - \Delta_2 = \Delta_3 - \Delta_4 \Leftrightarrow \Delta_1 + \Delta_4 = \Delta_2 + \Delta_3$). Then the following statement summarizes the results of Section 2.

Let M be the space of virtual convex bodies representable as a difference of convex bodies from the cone C_s. Then the functional F on C_s can be extended as an integral of the form $Q\omega$ over the chain of virtual convex bodies. Moreover, such an extension will be a polynomial on M.

In Section 3 we study the homological properties of unions X of (finite) arrangements of affine subspaces $\{L_i\}$ in a real Euclidean space $L = \mathbb{R}^n$ by means of the nerves K_X of their (closed) coverings by the sets L_i.

Given two affine subspace arrangements indexed by the same finite set of indices I, we say that the nerve K_X of the collection $\{L_i\}$ *dominates* the nerve K_Y of the collection $\{M_i\}$ if

$$\bigcap_{j \in J} L_j \neq \varnothing \quad \Rightarrow \quad \bigcap_{j \in J} M_j \neq \varnothing \qquad \forall J \subset I,$$

and we write $K_X \geq K_Y$ in this case. Furthermore, we say that a continuous map $f\colon X \to Y$ is *compatible* with K_X and K_Y if

$$x \in L_{i_1} \cap \ldots \cap L_{i_k} \quad \Rightarrow \quad f(x) \in M_{i_1} \cap \ldots \cap M_{i_k}.$$

Our main tool in the study of the homological properties of unions of affine subspaces is the following result.

(i) *If a map $f\colon X \to Y$ compatible with K_X and K_Y exists, then the condition $K_X \geq K_Y$ holds.*

(ii) *If a map $f\colon X \to Y$ compatible with K_X and K_Y exists, then it is unique up to homotopy.*

(iii) *If a nerve K_X is isomorphic to a nerve K_Y and a map $f\colon X \to Y$ compatible with K_X and K_Y exists, then f is a homotopy equivalence between X and Y.*

We then prove that any union X of affine subspaces has the so-called *good triangulation* (see Definition 3.6) and use this fact to show that if $K_X \geq K_Y$, then there is a map $f\colon X \to Y$ compatible with K_X and K_Y.

Now, suppose we have an arrangement of affine hyperplanes $\{H_i\}$ in $L = \mathbb{R}^n$. We call it *nondegenerate* if there is no proper linear subspace $V \subset \mathbb{R}^n$ which is parallel to all the hyperplanes H_i. Then the union X of such an arrangement has the homotopy type of a wedge of $(n-1)$-dimensional spheres, in which the number of spheres is equal to the number of bounded regions in $L \setminus X$ (see also Theorem 4.9). Therefore, each cycle $\Gamma \in H_{n-1}(X, \mathbb{Z})$ can be represented as a linear combination $\Gamma = \sum \lambda_j \partial \Delta_j$, where each coefficient λ_j equals the winding number of the cycle Γ around a point $a_j \in \Delta_j \setminus \partial \Delta_j$. Here, Δ_j denotes the closure of a bounded open polyhedron representing a bounded component of $L \setminus X$.

In Section 4 we study the homotopy properties of unions X of (finite) arrangements of affine subspaces $\{L_i\}$ in a real Euclidean space $L = \mathbb{R}^n$ by means of the methods developed in Section 3 and the theory of smooth convex bodies in the space L.

We say that two hyperplane arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ are *combinatorially equivalent* if the corresponding nerves $K_{\mathcal{H}_1}$ and $K_{\mathcal{H}_2}$ are isomorphic. Let $\mathcal{H} = \{H_1, \ldots, H_s\}$ and $\mathcal{H}' = \{H'_1, \ldots, H'_s\}$ be two combinatorially equivalent hyperplane arrangements, and let $X = \bigcup H_i$ and $Y = \bigcup H'_i$ be the corresponding unions of hyperplanes. Then there exists a canonical homotopy equivalence $f\colon X \to Y$. Moreover, we show that for any (finite) simplicial complex K there exists a (finite) affine subspace arrangement $\{L_i\}$ such that the nerve of the (closed) covering of X by the sets L_i is homotopy equivalent to X and has the homotopy type of the simplicial complex K.

In order to study the homotopy type of a union of affine subspaces in $\mathbb{R}^n$, we consider finite unions $U \subset \mathbb{R}^n$ of open convex bodies: $U = \bigcup U_i$; our goal is reduced to studying the homotopy type of the set $\mathbb{R}^n \setminus U$. We will do it making use of the following notion from convex geometry. By a *tail cone* $\mathrm{tail}(U)$ of a convex body U, we mean the set of points $v \in \mathbb{R}^n$ such that the inclusion $a + tv \in U$ holds for any $a \in U$ and $t \geq 0$.

It is easy to see that for any convex set $U \subset \mathbb{R}^n$ its tail cone $\mathrm{tail}(U)$ has the following properties:

- the set $\mathrm{tail}(U)$ is a convex closed cone in $\mathbb{R}^n$; a convex set U is bounded if and only if $\mathrm{tail}(U)$ is the origin $O \in \mathbb{R}^n$;

- if $\mathrm{tail}(U)$ is a vector space V, then for any transversal space V' (i.e., for any V' such that $\mathbb{R}^n = V \oplus V'$) the set U can be represented in the form $U = U' \oplus V$, where $U' = U \cap V'$ is a bounded convex set; that is, if $\mathrm{tail}(U)$ is a vector space, then one has $U = U' \oplus \mathrm{tail}(U)$ for a certain bounded convex set U'.

Our main result in Section 4 can be stated as follows. *The set $\mathbb{R}^n \setminus U$ is homotopy equivalent to the set $\mathbb{R}^n \setminus \bigcup \{a_i + \mathrm{tail}(U_i)\}$, where the summation is taken over all i such that $\mathrm{tail}(U_i)$ is a vector space.*

Now, assume that all the linear spaces $V_i = \mathrm{tail}(U_i)$ above are equal to the same linear space V and denote by T a subspace transversal to V, i.e., a linear subspace of $\mathbb{R}^n$ such that $\mathbb{R}^n = T \oplus V$. Then the set $\mathbb{R}^n \setminus U$ is homotopy equivalent to $T \setminus \{b_i\}$, where $b_i := T \cap \{a_i + V_i\}$. This statement completely describes the homotopy type of the set $\mathbb{R}^n \setminus \bigcup H_i$, where $\{H_i\}$ is any collection of affine

hyperplanes in $\mathbb{R}^n$. Indeed, the complement $\mathbb{R}^n \setminus \bigcup H_i$ is a union of open convex sets. Moreover, the maximal linear subspaces contained in $\operatorname{tail}(U_i)$ are the same for each U_i: each of them is equal to the intersection of the linear spaces $\widetilde{H}_i$ parallel to the affine hyperplanes H_i.

Volumes of generalized virtual polytopes and intersection rings of generalized quasitoric manifolds. In the second part of the paper we apply the theory of volume polynomials of generalized virtual polytopes to study the cohomology rings of generalized quasitoric manifolds.

First, we construct a special cellular structure for generalized quasitoric manifolds and deduce monomial and linear relations between characteristic submanifolds of codimension 2 in their intersection rings. Then we prove a topological version of the BKK theorem, based on the properties of the volume polynomial for a generalized virtual polytope, which yields a convex-theoretic formula for the self-intersection polynomial on the second cohomology of a generalized quasitoric manifold. Finally, we make use of the BKK theorem as well as the description of a Poincaré duality algebra worked out in [25, 16] to obtain a Pukhlikov–Khovanskii type description of the cohomology ring of a generalized quasitoric manifold.

In Section 5 we introduce the notion of a generalized virtual polytope and study the properties of integral functionals on the space of generalized virtual polytopes. Suppose Δ is a triangulation of an $(n-1)$-dimensional sphere on the vertex set $V(\Delta) = \{v_1, \ldots, v_m\}$. In what follows, we will identify a simplex of Δ with the set of its vertices viewed as a subset in $\{1, 2, \ldots, m\}$.

A map $\lambda \colon V(\Delta) \to (\mathbb{R}^n)^*$ is called a *characteristic map* if for any vertices $v_{i_1}, \ldots, v_{i_r}$ belonging to the same simplex of Δ the images $\lambda(v_{i_1}), \ldots, \lambda(v_{i_r})$ are linearly independent (over $\mathbb{R}$). Similarly, one can define the notion of an integer *characteristic map* $\lambda \colon V(\Delta) \to (\mathbb{Z}^n)^*$.

Such a map defines an m-dimensional family of hyperplane arrangements $\mathcal{AP}$ in the following way. For any $h = (h_1, \ldots, h_m) \in \mathbb{R}^m$, the arrangement $\mathcal{AP}(h)$ is given by

$$\mathcal{AP}(h) = \{H_1, \ldots, H_m\} \qquad \text{with} \quad H_i = \{\ell_i(x) = h_i\},$$

where we denote by ℓ_i the linear function $\lambda(v_i)$ for each $i \in [m] := \{1, 2, \ldots, m\}$. Given a subset $I \subset [m]$, we also define $H_I = \bigcap_{j \in I} H_j$. If $I \in \Delta$, then Γ_I denotes the face dual to I in the polyhedral complex $\Delta^\perp$ dual to the simplicial complex Δ. By definition, the facets of $\Delta^\perp$ are closed stars in Δ' of the vertices of Δ viewed as vertices of its barycentric subdivision Δ'.

By a *generalized virtual polytope* we mean a map $f \colon \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ *subordinate* to the characteristic map λ; that is, for any $I \subset [m]$, we have

$$f(\Gamma_I) \subset H_I.$$

Let U be a bounded region of $\mathbb{R}^n \setminus \bigcup_{\mathcal{AP}(h)} H_i$ and $W(U, f)$ be a winding number of a map f. Given a polynomial Q on $\mathbb{R}^n$, let us consider the following integral functional on the space of generalized virtual polytopes:

$$I_Q(f) := \sum W(U, f) \int_U Q\omega.$$

The key result of Section 5 is the computation of all partial derivatives of $I_Q(f)$, leading us to the following statement. *Let $I = \{i_1, \ldots, i_r\} \subseteq [m]$ be such that $I \notin \Delta$ and $k_1, \ldots, k_r$ be positive integers. Then we have*

$$\partial_{i_1}^{k_1} \ldots \partial_{i_r}^{k_r} (I_Q)(f) = 0.$$

However, if $r = n = \dim \Delta + 1$ and I is a simplex in Δ dual to a vertex $A \in \Delta^\perp$, then we have

$$\partial_I (I_Q)(f) = \operatorname{sgn}(I) Q(A) |\det(e_{i_1}, \ldots, e_{i_n})|.$$

We observe that the volume of the oriented image $f_h(\Delta^\perp) \subset \mathbb{R}^n$ is a function on the real vector space $\mathcal{L} = \{f_h \colon \Delta^\perp \to \mathbb{R}^n\}$, and its value $\mathrm{Vol}(f_h)$ on a generalized virtual polytope f_h is a homogeneous polynomial in $h_1, \ldots, h_m$ of degree n. This observation and the previous result yield the values of all the partial derivatives of order n for the volume polynomial $\mathrm{Vol}(f_h)$ of the generalized virtual polytope f_h and hence give us this homogeneous polynomial itself.

We start Section 6 by recalling the notion of a generalized quasitoric manifold introduced in [15]. In what follows we assume that $K = K_\Sigma$ is a *star-shaped sphere*, i.e., an intersection of a complete simplicial fan Σ in $\mathbb{R}^n \simeq N \otimes_{\mathbb{Z}} \mathbb{R}$ with the unit sphere $S^{n-1} \subset \mathbb{R}^n$. In this case, the moment–angle complex $\mathcal{Z}_K$ acquires a smooth structure (see [23]). Let further $\Lambda \colon \Sigma(1) \to N$ be a characteristic map. Then the $(m - n)$-dimensional subtorus $H_\Lambda := \ker \exp \Lambda \subset (S^1)^m$ acts freely on $\mathcal{Z}_K$, and the smooth manifold $X_{\Sigma,\Lambda} := \mathcal{Z}_K / H_\Lambda$ is called a *generalized quasitoric manifold*.

Our description of the cohomology of $X_{\Sigma,\Lambda}$ goes in three steps:

1. We provide a special cell decomposition for $X_{\Sigma,\Lambda}$ and show that $H^*(X_{\Sigma,\Lambda})$ is generated by the classes of characteristic submanifolds of codimension 2.

2. We deduce monomial and linear relations between classes of characteristic submanifolds of codimension 2 in $H_*(X_{\Sigma,\Lambda})$.

3. We prove a topological version of the BKK theorem for $X_{\Sigma,\Lambda}$ and then use it to get a Pukhlikov–Khovanskii type description of the intersection ring $H_*(X_{\Sigma,\Lambda})$.

It is worth mentioning that steps 2 and 3 above could be used in the much more general setting of torus manifolds. However, in this general case the algebra obtained by the Pukhlikov–Khovanskii description might be different from the intersection ring (cohomology ring). Indeed, the algebra (Theorem 6.11) computed via the self-intersection polynomial (Theorem 6.6) is the Poincaré duality quotient of the subalgebra of the cohomology ring generated by classes of characteristic submanifolds of codimension 2.

2. SMOOTH CONVEX BODIES AND THE SPACE OF MAPS $f \colon S^{n-1} \to \mathbb{R}^n$

In this section we consider a motivational construction of smooth virtual convex bodies.

Consider a set of smooth maps $f \colon S^{n-1} \to \mathbb{R}^n$. Such a set forms a vector space under scaling and pointwise addition of functions:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \qquad (\lambda f)(x) = \lambda f(x).$$

For a strictly convex smooth body $\Delta \subset \mathbb{R}^n$, its boundary $\partial \Delta$ can be identified with the image of the unit sphere under a Gauss map $f_\Delta \colon S^{n-1} \to \partial \Delta$.

In terms of the support function H_Δ of Δ, the map f_Δ is equal to the restriction of the gradient $\mathrm{grad}\, H_\Delta$ to the sphere S^{n-1}. Thus, we get an inclusion of the space of strictly convex smooth bodies (and their formal differences) into the space of smooth maps from S^{n-1} to $\mathbb{R}^n$. This inclusion respects the Minkowski addition of convex bodies.

We will be interested in integral functionals on the space of convex bodies. First notice that one can express the integral $\int_\Delta \omega$ in terms of the corresponding map f_Δ:

$$\int_\Delta \omega = \int_{S^{n-1}} f^*\alpha,$$

where α is any form such that $d\alpha = \omega$.

Let α be an $(n-1)$-form on $\mathbb{R}^n$ given by

$$\alpha = P_1 \, \widehat{dx_1} \wedge \ldots \wedge dx_n + \ldots + P_n \, dx_1 \wedge \ldots \wedge \widehat{dx_n}.$$

Here, $\widehat{dx_i}$ means that the term dx_i is missing. The following theorem is obvious.

Theorem 2.1. *If all coefficients P_i of the form α are polynomials of degree at most k on $\mathbb{R}^n$, then the function $\int_{S^{n-1}} f^*\alpha$ on the space of smooth maps $f\colon S^{n-1} \to \mathbb{R}^n$ is a polynomial of degree at most $k+n-1$.*

If all coefficients P_i of the form α are homogeneous polynomials of degree k, then the function $\int_{S^{n-1}} f^*\alpha$ is a homogeneous polynomial of degree $k+n-1$ on the space of smooth maps.

Integral functional on the space of maps and winding numbers. For an $(n-1)$-form α on $\mathbb{R}^n$ and a smooth map $f\colon S^{n-1} \to \mathbb{R}^n$, one can use a different way to compute the integral $\int_{S^{n-1}} f^*\alpha$. Let $U \subset \mathbb{R}^n$ be a connected component of $\mathbb{R}^n \setminus f(S^{n-1})$.

Definition 2.2. The *winding number* $W(U, f)$ of U with respect to f is the mapping degree of the map

$$\frac{f-a}{\|f-a\|}\colon\ S^{n-1} \to S^{n-1}, \tag{2.1}$$

where a is any point in U.

The mapping degree is well defined, i.e., is independent of the choice of $a \in U$, since the maps (2.1) for different $a \in U$ are homotopic to each other.

Proposition 2.3. *For any smooth $(n-1)$-form α on $\mathbb{R}^n$ and any smooth map $f\colon S^{n-1} \to \mathbb{R}^n$, the following identity holds:*

$$\int_{S^{n-1}} f^*\alpha = \sum W(U, f) \int_U d\alpha,$$

where the sum is taken over all connected components U of the complement $\mathbb{R}^n \setminus f(S^{n-1})$.

Proof. This follows from the Stokes formula. $\square$

Theorem 2.4. *Let Q be a polynomial of degree at most k (homogeneous polynomial of degree k) on $\mathbb{R}^n$ and let $\omega = dx_1 \wedge \ldots \wedge dx_n$ be the standard volume form on $\mathbb{R}^n$. Then the function*

$$\sum W(U, f) \int_U Q\omega$$

on the space of smooth maps is a polynomial of degree at most $k+n$ (homogeneous polynomial of degree $k+n$).

Proof. Consider an $(n-1)$-form $\alpha = P\,dx_2 \wedge \ldots \wedge dx_n$, where P is a polynomial of degree at most $k+1$ such that $\partial P/\partial x_1 = Q$. Clearly, $d\alpha = Q\omega$. Thus the statement follows from Theorem 2.1 and Proposition 2.3. $\square$

Let us denote by C_s the cone of strictly convex bodies $\Delta \subset \mathbb{R}^n$ with smooth boundaries. As a corollary, we obtain the following result.

Corollary 2.5. *Let Q and ω be the same as before. Then the function*

$$F(\Delta) = \int_\Delta Q\omega \tag{2.2}$$

on the cone C_s is a polynomial of degree at most $k+n$ (homogeneous polynomial of degree $k+n$).

Proof. Indeed, for the map $f = \operatorname{grad} H_\Delta\colon S^{n-1} \to \mathbb{R}^n$ there are exactly two connected components of $\mathbb{R}^n \setminus f(S^{n-1})$: the component $U_1 = \mathbb{R}^n \setminus \Delta$ and the component $U_2 = \operatorname{int}(\Delta)$. The corresponding winding numbers are $W(U_1, f) = 0$ and $W(U_2, f) = 1$. Thus, the statement follows from Theorem 2.4. $\square$

We would like to extend the integral functional to the vector space generated by the cone C_s.

Definition 2.6. 1. A *virtual convex body* is a formal difference of two convex bodies (with the usual identification $\Delta_1 - \Delta_2 = \Delta_3 - \Delta_4 \Leftrightarrow \Delta_1 + \Delta_4 = \Delta_2 + \Delta_3$).

2. A *support function of a virtual convex body* $\Delta = \Delta_1 - \Delta_2$ is the difference of the support functions of Δ_1 and Δ_2.

3. A *chain of virtual convex bodies with a smooth support function H* is the set of connected components U of the complement $\mathbb{R}^n \setminus \operatorname{grad} H(S^{n-1})$ taken with the coefficients $W(U, \operatorname{grad} H)$.

The following theorem summarizes the results of this section.

Theorem 2.7. *Let M be the space of virtual convex bodies representable as differences of convex bodies from the cone C_{s}. Then the function* (2.2) *on C_{s} can be extended to M as an integral of the form $Q\omega$ over the chain of virtual convex bodies. Moreover, such an extension is given by a polynomial on M.*

3. UNIONS OF AFFINE SUBSPACES

In this section we study homological properties of unions of (finite) affine subspace arrangements in a vector space $L \simeq \mathbb{R}^n$. Let I be a finite set of indices. Consider a set $\{L_i\}$ of affine subspaces in L indexed by elements $i \in I$, and let $X = \bigcup_{i \in I} L_i$ be their union.

First, we define the main combinatorial invariant of a union of a collection of affine subspaces. Note that the topological space X has a natural covering by the affine subspaces L_i.

Definition 3.1. The *nerve K_X* of the natural covering of X is the simplicial complex with vertex set indexed by I (i.e., one vertex for each index $i \in I$). A set of vertices $v_{i_1}, \ldots, v_{i_k}$ defines a simplex in K_X if and only if the intersection $L_{i_1} \cap \ldots \cap L_{i_k}$ is not empty.

Consider another collection of affine subspaces $\{M_i\}$ in a vector space M indexed by the same set I, with a nerve K_Y corresponding to the natural covering of their union Y.

Definition 3.2. We will say that the nerve K_X of the collection $\{L_i\}$ *dominates* the nerve K_Y of the collection $\{M_i\}$ if

$$\bigcap_{j \in J} L_j \neq \varnothing \quad \Rightarrow \quad \bigcap_{j \in J} M_j \neq \varnothing \qquad \forall\, J \subset I.$$

We will write $K_X \geq K_Y$ in this case.

We say that the nerves K_X and K_Y are *equivalent* if $K_X \geq K_Y$ and $K_Y \geq K_X$.

Note that if $K_X \geq K_Y$, then there is a natural inclusion $K_X \to K_Y$. Moreover, if K_X and K_Y are equivalent, then this inclusion provides an isomorphism between these complexes.

3.1. Maps compatible with coverings. In this subsection we introduce our main tool in the study of unions of affine subspace arrangements. Let, as before, $X = \bigcup_{i \in I} L_i$ and $Y = \bigcup_{i \in I} M_i$ be two collections of affine subspaces indexed by a finite set I. First, we will need the following definition.

Definition 3.3. For a point $x \in X = \bigcup_{i \in I} L_i$, let $I(x)$ be the subset of indices in I such that

$$x \in L_i \quad \Leftrightarrow \quad i \in I(x).$$

For two points $x \in X$ and $y \in Y$, we write $x \geq y$ if $I(x) \supset I(y)$.

In particular, Definition 3.3 leads to the following notion.

Definition 3.4. A continuous map $f\colon X \to Y$ is *compatible* with K_X and K_Y if $x \leq f(x)$ for any $x \in X$, or, in other words, if

$$x \in L_{i_1} \cap \ldots \cap L_{i_k} \quad \Rightarrow \quad f(x) \in M_{i_1} \cap \ldots \cap M_{i_k}.$$

The following theorem is our main tool in the study of homological properties of unions of affine subspace arrangements.

Theorem 3.5. *The following statements hold*:

(i) *if a map $f\colon X \to Y$ compatible with K_X and K_Y exists, then the condition $K_X \geq K_Y$ holds*;

(ii) *if a map $f\colon X \to Y$ compatible with K_X and K_Y exists, then it is unique up to homotopy*;

(iii) *if K_X is isomorphic to K_Y, then the map $f\colon X \to Y$ compatible with K_X and K_Y provides a homotopy equivalence between X and Y.*

Proof. (i) Assume a map $f\colon X \to Y$ compatible with K_X and K_Y exists. Then $K_X \geq K_Y$. Indeed, if the set $L_{i_1} \cap \ldots \cap L_{i_k}$ is not empty and contains a point x, then the set $M_{i_1} \cap \ldots \cap M_{i_k}$ contains $f(x)$ and, in particular, is nonempty.

(ii) If f and g are two maps from X to Y compatible with K_X and K_Y, then for any $0 \leq t \leq 1$ the map $tf + (1-t)g$ is also compatible with K_X and K_Y. Indeed, for any $x \in X$, the set of points $y \in Y$ such that $I(x) \subset I(y)$ is convex.

(iii) Assume that K_X and K_Y are isomorphic and there are maps $f\colon X \to Y$ and $g\colon Y \to X$ compatible with K_X and K_Y. Then the map $g \circ f\colon X \to X$ is a homotopy equivalence. Indeed, the identity map Id_X and the composition map $g \circ f$ are compatible with K_X and hence are homotopy equivalent by statement (ii). Similarly, the composition map $f \circ g\colon Y \to Y$ is homotopic to the identity map Id_Y. $\square$

To prove the existence of compatible maps, we will need the following definitions.

Definition 3.6. A *good triangulation* of the set $X = \bigcup_{i \in I} L_i$ is a triangulation such that the following condition holds: the vertex set of a simplex S in a good triangulation is totally ordered in the sense of Definition 3.3. In other words, there is an order on the vertex set $\{v_{i_1}, \ldots, v_{i_s}\}$ of S such that

$$I(v_{i_1}) \subset \ldots \subset I(v_{i_s}).$$

Definition 3.7. Consider the following *natural stratification* of $X = \bigcup_{i \in I} L_i$ by open strata of different dimensions: we say that two points $x, y \in X$ belong to one stratum if $x \geq y$ and $y \geq x$, or equivalently, $I(x) = I(y)$. The stratum containing the point x is the intersection $L(x)$ of the subspaces L_i for all $i \in I(x)$ with the removed union of the subspaces L_i for all $i \notin I(x)$.

Definition 3.8. A stratum U_1 of the natural stratification of X is *bigger* than a stratum U_2 of the same stratification $(U_1 \geq U_2)$ if the closure of U_1 contains U_2.

It is easy to see that $U_1 \geq U_2$ if and only if the relation $x \geq y$ holds for any $x \in U_1$ and $y \in U_2$.

Definition 3.9. A stratum U has *rank k* if the longest possible chain of strictly decreasing strata $U = U_1 > \ldots > U_k$ has length k.

Theorem 3.10. *For any finite union $X = \bigcup L_i$ of affine subspaces L_i in a linear space L, one can construct a good triangulation of X.*

Proof. We construct a good triangulation for X in two steps. First, we construct a triangulation compatible with the natural stratification of X, i.e., a triangulation such that any open simplex is contained in a certain open stratum.

A triangulation compatible with the natural stratification of X can be constructed inductively by first triangulating all strata of rank 1 (i.e., all closed strata) and then extending it to all strata of rank higher by one at each step.

Then one can construct a good triangulation for X by taking a barycentric subdivision of any triangulation of X compatible with the standard (natural) stratification. Indeed, the set of vertices of each simplex in this subdivision corresponds to an increasing chain of faces of a simplex in the original triangulation, which are contained in an increasing chain of strata. $\square$

Theorem 3.11. *If $K_X \geq K_Y$, then there exists a map $f\colon X \to Y$ compatible with K_X and K_Y.*

Proof. First, let us consider a good triangulation τ of X. Then, for any vertex v of τ, let us define the value $f(v)$ to be any point in Y such that $I(f(x)) \supset I(x)$. Such a point always exists since $K_X \geq K_Y$. Then we can extend the map f linearly to each simplex of τ.

The map f constructed above is compatible with K_X and K_Y. Indeed, for any point $x \in X$, there exists a smallest simplex S of the good triangulation for X such that $x \in S$. Among the vertices $V(S)$ of this simplex S, there exists a biggest vertex v. It is easy to see that $I(x) = I(v)$. Since $f(x)$ belongs to the linear combination of the points $f(v_i)$ with $v_i \in V(S)$, the inclusion $I(f(x)) \supset I(x)$ holds. $\quad\square$

3.2. Barycentric subdivision and a covering of a simplicial complex. We will need some general facts related to barycentric subdivisions of simplicial complexes.

Let C' be the simplicial complex obtained by the barycentric subdivision of a given simplicial complex C. Each vertex of C' is the barycenter of a certain simplex of C. A set of vertices of C' belongs to one simplex of C' if and only if the simplices of C corresponding to these vertices are totally ordered with respect to inclusion.

With each vertex v of C let us associate the closed subset X_v of C' equal to the union of all simplices of C' containing the vertex v.

Lemma 3.12. 1. *The nerve of the covering of C' by the collection of closed subsets X_v corresponding to all vertices v of C coincides with the original complex C.*

2. *All sets X_v and their nonempty intersections are homotopy equivalent to a point.*

Proof. 1. By definition, the set of vertices v of C can by identified with the set of subsets X_v, which provides a covering of C'. If vertices $v_1, \ldots, v_k$ belong to one simplex of C, then the sets $X_{v_1}, \ldots, X_{v_k}$ contain the barycenter of that simplex, and so these sets have a nonempty intersection.

Conversely, a set X_v intersects a simplex Δ of the complex C only if v is a vertex of Δ. Thus, if the intersection $X_{v_1} \cap \ldots \cap X_{v_k}$ is not empty, then $v_1, \ldots, v_k$ belong to a simplex Δ of C.

2. Any nonempty intersection $X_{v_1} \cap \ldots \cap X_{v_k}$ can be represented as a union of some simplices of C' containing a common vertex, which is the barycenter of the simplex with the vertices $v_1, \ldots, v_k$. Observe that such a union is a cone; hence it is homotopy equivalent to a point. $\quad\square$

3.3. Maps $f\colon K'_X \to Y$ in the case $K_X \geq K_Y$. A continuous map $f\colon K'_X \to Y$ is compatible with the natural coverings

$$BK_X = \bigcup_{i \in I} X_{v_i} \qquad \text{and} \qquad Y = \bigcup_{i \in I} M_i$$

if the inclusion $f(X_{v_i}) \subset M_i$ holds for any $i \in I$.

Suppose that $K_X \geq K_Y$. Let K_X be the nerve of the natural covering of $X = \bigcup_{i \in I} L_i$. The barycentric subdivision K'_X of K_X has its own natural covering by the sets $\widehat{L}_i$ equal to the unions of the simplices in K'_X which contain the vertex v_i corresponding to the space L_i. By Lemma 3.12, the nerve of this covering of K'_X is isomorphic to K_X. Now, let us generalize the definition of a map between topological spaces that is compatible with their coverings. Suppose I is a finite set of indices. Consider a set $\{X_i\}$ of closed subsets of X indexed by elements $i \in I$.

Definition 3.13. The *nerve K_X* of the covering $X = \bigcup X_i$ is the simplicial complex whose vertex set V_X contains one vertex v_i for each subset X_i, i.e., one vertex for each index $i \in I$. A set of vertices $v_{i_1}, \ldots, v_{i_k}$ defines a simplex in K_X if and only if the intersection $X_{i_1} \cap \ldots \cap X_{i_k}$ is not empty.

The following theorem can be proved in exactly the same way as Theorem 3.5.

Theorem 3.14. 1. *A map $f\colon K'_X \to Y$ compatible with K_X and K_Y exists if and only if the condition $K_X \geq K_Y$ holds.*

2. *If a map compatible with K_X and K_Y exists, then it is unique up to homotopy.*

4. HOMOTOPY TYPE OF A UNION OF AN AFFINE SUBSPACE ARRANGEMENT

In this section we study the homotopy type of a union of an affine subspace arrangement. In particular, we show that a union of a collection of affine subspaces can have a homotopy type of any simplicial complex (Theorem 4.4), whereas a union of affine hyperplanes is always homotopic to a wedge of spheres (Theorem 4.9).

Consider a finite set $\{A_i\}$ of affinely independent points in a real vector space L. Let $T \subset L$ be the simplex on the vertex set $\{A_i\}$. Along with each face T_J of T, consider the affine hull L_{T_J} of T_J. We obtain a collection of affine subspaces in L corresponding to the faces T_J.

Recall that a subspace A of a topological space X is called a *strong deformation retract* of X if there is a homotopy $\pi(x,t)\colon X \times I \to X$ such that

(i) $\pi(x,0) = x$ for any $x \in X$;

(ii) $\pi(x,1) \in A$ for any $x \in X$;

(iii) $\pi(a,t) = a$ for any $a \in A$ and $t \in I$.

Lemma 4.1. *The simplex T is a strong deformation retract of the union of hyperplanes in L. Moreover, the deformation retraction $\pi\colon L \times I \to L$ can be chosen to preserve the covering of L by the affine subspaces L_{T_i}, i.e.,*

$$\pi(x,t) \in L_{T_i} \qquad \text{for any} \quad x \in L_{T_i}, \quad t \in I.$$

Proof. Note that each point $x \in L$ is representable in a unique way as

$$x = \sum \lambda_i A_i, \qquad \text{where} \quad \sum \lambda_i = 1$$

(the numbers λ_i are the *barycentric coordinates* of x with respect to the simplex T).

Consider the projection $p\colon L \to T$ that maps a point x with barycentric coordinates $\{\lambda_i\}$ to the point $p(x)$ whose ith barycentric coordinate is equal to $\max\{\lambda_i, 0\}$.

It is easy to see that the map $\pi(x,t)$ defined by

$$\pi(x,t) = (1-t)x + tp(x)$$

satisfies the conditions of the lemma. $\square$

Let $\{T_i\}$ be an ordered collection of faces of the simplex T of size N. Consider the following two sets, each equipped with a covering by N closed convex sets:

- the union $\bigcup_{i=1}^N T_i$, equipped with the covering by the faces T_i from the set $\{T_i\}$;
- the union $\bigcup_{i=1}^N L_{T_i}$ of the affine hulls L_{T_i} of the faces T_i, equipped with the covering by the spaces L_{T_i}.

Theorem 4.2. *The natural embedding $\bigcup T_i \to \bigcup L_{T_i}$ makes $\bigcup T_i$ into a strong deformation retract of $\bigcup L_{T_i}$. Moreover, the deformation retraction can be chosen to preserve the covering of $\bigcup L_{T_i}$ by the affine spaces L_{T_i}.*

Proof. Indeed, as the required projection and its homotopy one can take the restriction of the homotopy from Lemma 4.1 to the space $\bigcup L_{T_i}$. $\square$

4.1. Barycentric subdivision and the corresponding affine subspaces. Let Δ be a simplicial complex and let Δ' be its barycentric subdivision. In particular, any simplex Δ_i in Δ corresponds to a vertex A_{Δ_i} of Δ'.

In a vector space L, consider a collection of affinely independent points identified with the vertices of Δ'. Then Δ' is naturally embedded in the simplex T generated by this collection.

For a vertex A_i of Δ, let its *star* $\operatorname{St}(A_i)$ be the collection of simplices of Δ having A_i as a vertex. Each star $\operatorname{St}(A_i)$ determines a face T_i of T; it is the convex hull of the vertices of T that correspond to the simplices in $\operatorname{St}(A_i)$.

Let X_Δ be the union of all faces $T_i \subset T$ corresponding to the vertices of Δ. Then the space $X = X_\Delta$ has a natural covering by the faces T_i. On the other hand, let Y be the union of the affine hulls L_{T_i} of the faces T_i corresponding to the vertices of Δ. Then the space Y has a natural covering by the subspaces L_{T_i}.

The following statement is an immediate corollary of Theorem 4.2.

Corollary 4.3. *The subset $X \subset Y$ is a deformation retract of Y. Moreover, the deformation retraction respects the coverings of X by T_i and of Y by L_{T_i}.*

Theorem 4.4. *The nerve of the covering of X by T_i can be naturally identified with the nerve of the covering of Y by L_{T_i}. Both of these nerves can be naturally identified with the original simplicial complex Δ.*

We can consider the barycentric subdivision Δ' of Δ as a subcomplex of the complex of all faces of the simplex T. Denote by Z the union of all simplices in Δ'. The space Z is equipped with the following covering: with each vertex A_i of Δ one can associate the union Z_i of all (closed) simplices containing the vertex A_i. In other words, Z_i is the union of all faces of T that contain the vertex A_i and belong to the simplicial complex Δ'.

Observe that under the embedding $Z \to X$ the sets Z_i are identified with $T_i \cap Z$.

Theorem 4.5. *There exists a map $\pi\colon X \to Z$ such that the following conditions hold*:

(1) π *maps each simplex T_i to the set Z_i*;

(2) π *maps each simplex from Z to itself.*

Proof. The set X is stratified by its covering $X = \bigcup T_i$ in the following way. Each stratum of this stratification is a nonempty intersection of a certain collection of the sets T_i without all nonempty intersections of the bigger collections of sets T_i. In particular, this stratification also stratifies the set $Z \subset X$.

The set of all strata of the above stratification can be naturally identified with the set of all simplices of Δ. Indeed, the intersection $\bigcap T_{i_j}$ is nonempty if and only if there is a simplex in Δ with the vertices A_{i_j}.

In other words, the set of all strata is in one-to-one correspondence with the set of vertices of Δ', i.e., with the set of vertices of T.

The triangulation of X by the faces of T belonging to X is compatible with the above stratification; i.e., each open simplex of this triangulation is contained in a certain stratum.

Consider the barycentric subdivision of the triangulation constructed above. Note that it provides a good triangulation for our stratification; i.e., each simplex from this triangulation is compatible in the following sense: if two strata contain two vertices of a simplex of the triangulation, then one of the strata belongs to the closure of the other.

Now we are ready to define a map π. The map π is a map from X to Z which is linear on each simplex of the barycentric subdivision of the natural triangulation of X and maps each vertex A of the triangulation to the vertex of Δ' corresponding to the stratum containing the vertex A.

One can easily check that the map just constructed satisfies all conditions of the theorem. $\square$

Theorem 4.6. *The map $\pi\colon X \to Z \subset X$ is homotopic to the identity map. Denote by $\widetilde{\pi}$ the restriction of π to Z. Then $\widetilde{\pi}$ maps Z to itself and this map is homotopic to the identity map.*

Proof. Observe that if $x \in T_i \subseteq X$, then $\pi(x)$ also belongs to T_i, as well as the entire segment joining these two points, due to the definition of the map π. Therefore, one can define a linear homotopy $F(x,t) = (1-t)x + t\pi(x)$ between the identity map and the map π.

Furthermore, $\widetilde{\pi}$ maps each simplex of Δ' to itself. Hence one can define a linear homotopy $G(x,t) = (1-t)x + t\widetilde{\pi}(x)$ between the identity map and the map π. $\quad\square$

4.2. Homotopy type of a union of a hyperplane arrangement. Let $\mathcal{H} = \{H_1, \ldots, H_s\}$ be a collection of affine hyperplanes in $L \simeq \mathbb{R}^n$ indexed by the set $[s] = \{1, \ldots, s\}$.

Definition 4.7. The *nerve* $K_{\mathcal{H}}$ of $\mathcal{H}$ is the simplicial complex on s vertices $v_1, \ldots, v_s$ such that a set of vertices $v_{i_1}, \ldots, v_{i_k}$ defines a simplex in $K_{\mathcal{H}}$ if and only if the intersection $H_{i_1} \cap \ldots \cap H_{i_k}$ is not empty.

We will say that two hyperplane arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ are *combinatorially equivalent* if the corresponding nerves $K_{\mathcal{H}_1}$ and $K_{\mathcal{H}_2}$ are isomorphic.

Theorem 4.8. *Suppose $\mathcal{H} = \{H_1, \ldots, H_s\}$ and $\mathcal{H}' = \{H_1', \ldots, H_s'\}$ are two combinatorially equivalent hyperplane arrangements, and let $X = \bigcup H_i$ and $Y = \bigcup H_i'$ be the corresponding unions of hyperplanes. Then there exists a canonical homotopy equivalence $f\colon X \to Y$.*

Proof. As a canonical homotopy equivalence $f\colon X \to Y$, one can take any continuous map such that

$$f(x) \in H_j' \qquad \text{for every} \quad x \in H_j. \qquad \square$$

In particular, there is a canonical isomorphism $f_*\colon H_*(X) \to H_*(Y)$ between the homology groups of combinatorially equivalent hyperplane arrangements.

We will say that a collection of hyperplanes $\{H_1, \ldots, H_s\}$ is *nondegenerate* if it is a nondegenerate collection of affine subspaces; that is, there is no proper linear subspace $L \subset \mathbb{R}^n$ parallel to all the hyperplanes H_i.

Theorem 4.9. *Let $\mathcal{H}$ be a nondegenerate arrangement of affine hyperplanes in $\mathbb{R}^n$. Then its union X is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres. The number of spheres is equal to the number of bounded regions in $\mathbb{R}^n \setminus X$.*

We will prove a more general result (see Theorem 4.13 and Corollary 4.14).

Corollary 4.10. *Let $L \supset X = \bigcup L_i$ be a nondegenerate union of affine hyperplanes L_i. Then, if $n > 1$, the group $H_{n-1}(X, \mathbb{Z})$ is a free abelian group generated by the cycles $\partial\Delta_j$, where Δ_j is the closure of the bounded open polyhedron representing a bounded component of $L \setminus X$.*

All the other groups $H_i(X, \mathbb{Z})$ for $i > 0$ are equal to zero, and $H_0(X, \mathbb{Z}) \cong \mathbb{Z}$.

According to Corollary 4.10, every cycle $\Gamma \in H_{n-1}(X, \mathbb{Z})$ can be represented as a linear combination

$$\Gamma = \sum \lambda_j\, \partial\Delta_j.$$

Moreover, each coefficient λ_j equals the winding number of the cycle Γ around a point $a_j \in \Delta_j \setminus \partial\Delta_j$.

Corollary 4.11. *Suppose $L^0 \subset L$ is a linear space such that $L = L^0 \oplus \widehat{L}$, i.e., L is a direct sum of $\widehat{L}$ and L^0. Let $L_i^0 = L_i \cap L^0$ and $X^0 = X \cap L^0 = \bigcup L_i^0$. Then X^0 is a union of a nondegenerate arrangement of the affine hyperplanes $L_i^0 \subset L^0$. Moreover, $X = X^0 \times \widehat{L}$; thus X is homotopy equivalent to a wedge of $(n-1-l)$-dimensional spheres, where $l = \dim \widehat{L}$.*

Now we are ready to prove Theorem 4.9. Let $U \subset \mathbb{R}^n$ be a finite union of open convex bodies: $U = \bigcup U_i$. We are going to study the homotopy type of the set $\mathbb{R}^n \setminus U$. First, we need the following definition.

Definition 4.12. The *tail cone* of a convex body U is the set of points $v \in \mathbb{R}^n$ such that the inclusion $a + tv \in U$ holds for any $a \in U$ and $t \geq 0$.

One can check that for any convex set $U \subset \mathbb{R}^n$ the set $\mathrm{tail}(U)$ satisfies the following conditions:

- the set $\mathrm{tail}(U)$ is a convex closed cone in $\mathbb{R}^n$; a convex set U is bounded if and only if $\mathrm{tail}(U)$ is the origin $O \in \mathbb{R}^n$;
- if $\mathrm{tail}(U)$ is a vector space L, then for any transversal space L_1 (i.e., for any L_1 such that $\mathbb{R}^n = L \oplus L_1$) the set U is representable in the form $U = U_1 \oplus L$, where $U_1 = U \cap L_1$ is a bounded convex set; that is, if $\mathrm{tail}(U)$ is a vector space, then one has $U = U_1 \oplus \mathrm{tail}(U)$ for a certain bounded convex set U_1.

If the set $\mathrm{tail}(U_i)$ is a linear space L_i, then along with U_i we can also consider a shifted space $a_i + L_i \subset U_i$, where a_i is an arbitrary point in U_i.

We will prove the following theorem.

Theorem 4.13. *For the set U defined above, the set $\mathbb{R}^n \setminus U$ is homotopy equivalent to the set $\mathbb{R}^n \setminus \bigcup \{a_i + L_i\}$, where the union is taken over all indices i such that $\mathrm{tail}(U_i)$ is a vector space.*

Suppose that in the above theorem all the linear spaces L_i are equal to the same linear space L. Denote by T a transversal subspace to L, i.e., a linear subspace in $\mathbb{R}^n$ such that $\mathbb{R}^n = T \oplus L$.

Corollary 4.14. *Under the above assumptions, the set $\mathbb{R}^n \setminus U$ is homotopy equivalent to $T \setminus \{b_i\}$ with $b_i = T \cap \{a_i + L_i\}$.*

Note that Corollary 4.14 completely describes the homotopy type of the set $\mathbb{R}^n \setminus \bigcup H_i$, where $\{H_i\}$ is any collection of affine hyperplanes in $\mathbb{R}^n$. Indeed, the complement $\mathbb{R}^n \setminus \bigcup H_i$ is a union of open convex sets. Moreover, the maximal linear subspaces contained in $\mathrm{tail}(U_i)$ are the same for each U_i: each of them is equal to the intersection of all linear spaces $\widetilde{H}_i$ parallel to the affine hyperplanes H_i.

To prove the theorem, we will need some general facts about convex bodies.

Lemma 4.15. *Suppose $U \subset \mathbb{R}^n$ is a bounded open convex set, X is the closure of U, and ∂X is the boundary of X (one has $X = U \cup \partial X$), and let $a \in U$ be any point in U. Then ∂X is a deformation retract of $X \setminus \{a\}$.*

Proof. Let $\pi \colon X \setminus \{a\} \to \partial X$ be the projection of $X \setminus \{a\}$ to ∂X from the point a. The following map provides a deformation retraction:

$$F(x, t) = (1 - t)x + t\pi(x), \qquad \text{where} \quad x \in X \setminus \{a\}, \quad 0 \leq t \leq 1. \qquad \square$$

Corollary 4.16. *Let $U \subset \mathbb{R}^n$ be an open convex set such that $\mathrm{tail}(U)$ is a vector space L. Then, by definition, for any $a \in U$ the shifted space $a + L$ belongs to U and the set X is homotopy equivalent to the set $X \setminus L$, where X is the closure of U.*

We will need the following auxiliary lemma. Let us represent $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \oplus \mathbb{R}^1$, and let us accordingly use the notation (x, y) for points in $\mathbb{R}^n$, where $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}^1$.

Let $y = f(x)$ be a continuous function on $\mathbb{R}^{n-1}$. Denote by $X \subset \mathbb{R}^n$ the set of points (x, y) with $y \geq f(x)$. Then ∂X is the graph of the function f (i.e., $(x, y) \in \partial X$ if and only if $y = f(x)$).

Lemma 4.17. *The natural projection $\pi \colon X \to \partial X$ mapping a point (x, y) to $(x, f(x))$ is homotopic to the identity map.*

Proof. One can consider the following homotopy:

$$G(x, y, t) = (1 - t)(x, y) + t\pi(x, y). \qquad \square$$

Now, suppose that the set $\mathrm{tail}(U) \subset \mathbb{R}^n$ is not a vector space; i.e., assume that there is a vector $v \in \mathrm{tail}(U)$ such that the vector $-v$ does not belong to $\mathrm{tail}(U)$.

Let $a \in U$ be an arbitrary point. Since $-v$ is not in $\mathrm{tail}(U)$, there is a positive number τ such that $a - \tau v \in \partial X$. Let $\widetilde{L}$ be the supporting hyperplane of X at the point $a - \tau v$.

Let us make an affine change of variables in $\mathbb{R}^n$ in such a way that the hyperplane $\widetilde{L}$ becomes the hyperplane $y = 1$, the point $a - \tau v$ becomes the point $(0, 1)$, and the vector v becomes the standard basis vector in $\mathbb{R}^1$. After this change of coordinates, U turns into an open convex set in $\mathbb{R}^{n-1} \oplus \mathbb{R}^1$ such that U belongs to the half-space $y \geq 1$, and along with every point $a \in U$ our convex set U contains the entire ray $a + \tau v$, where $\tau \geq 0$ and v is the vector $(0, 1)$.

Consider the diffeomorphism g of the open half-space $y > 0$ to itself defined by the formula $g(x, y) = (xy, y)$.

Lemma 4.18. *Under the diffeomorphism g, the closure X of U is mapped to the domain Y defined by the following condition: $(x, y) \in Y$ if and only if $y \geq f(x)$, where f is a certain continuous function on $\mathbb{R}^n$.*

Proof. First, let us consider the map $\widetilde{g}\colon \partial X \to \mathbb{R}^{n-1} \oplus \{0\}$ defined by $\widetilde{g}(x, y) = (xy, 0)$. Let us show that $\widetilde{g}$ is a homeomorphism between ∂X and $\mathbb{R}^{n-1}$. For each vector $x \in \mathbb{R}^{n-1} \oplus \{0\}$, consider the set of points $\partial X_x \subset \partial X$ defined by the following condition: $(x_0, y_0) \in \partial X_x$ if and only if x_0 is proportional to x. It is easy to see that the set ∂X_x is homeomorphic to a line.

We can parametrize it by an oriented distance from the point $(0, 1)$ (which belongs to ∂X_x for every x) along this curve with an arbitrarily chosen orientation.

Now $\widetilde{g}$ maps the curve ∂X_x to the line of scalar multiples of x. Moreover, this map is monotonic and proper. Hence it provides a homeomorphism between ∂X_x and the line τx, $\tau \in \mathbb{R}$. This argument implies that the map $\widetilde{g}\colon \partial X \to \mathbb{R}^{n-1}$ is a homeomorphism.

Observe that the image of ∂X under the diffeomorphism $g\colon (x, y) \mapsto (xy, y)$ is a graph of the function f such that the value $f(x)$ equals the coordinate y of the point $(x, y) := \widetilde{g}^{-1}(x)$. Then the set X is mapped by this diffeomorphism to the domain in $\mathbb{R}^n$ consisting of the points (x, y) with $y \geq f(x)$. $\square$

Corollary 4.19. *Let $U \subset \mathbb{R}^n$ be an open convex set such that the cone $\mathrm{tail}(U)$ is not a vector space. Then the boundary ∂X of the closure X of U is homotopy equivalent to X.*

5. GENERALIZED VIRTUAL POLYTOPES: DEFINITION AND RESULTS

Let Δ be a simplicial complex homeomorphic to the $(n-1)$-dimensional sphere. Denote by $V(\Delta) = \{v_1, \ldots, v_m\}$ the set of vertices of Δ. In what follows, we will identify a simplex S of Δ with the set of vertices $I \subset V(\Delta)$ which belong to S.

Definition 5.1. The map $\lambda\colon V(\Delta) \to (\mathbb{R}^n)^*$ is called a *characteristic map* if for any vertices $v_{i_1}, \ldots, v_{i_r}$ that belong to the same simplex of Δ the images $\lambda(v_{i_1}), \ldots, \lambda(v_{i_r})$ are linearly independent. In particular, for any maximal simplex $\{v_{i_1}, \ldots, v_{i_n}\}$ the images $\lambda(v_{i_1}), \ldots, \lambda(v_{i_n})$ form a basis of $(\mathbb{R}^n)^*$.

The map $\lambda\colon V(\Delta) \to (\mathbb{Z}^n)^*$ is called an integer *characteristic map* if for any maximal simplex $\{v_{i_1}, \ldots, v_{i_n}\}$ of Δ the images $\lambda(v_{i_1}), \ldots, \lambda(v_{i_n})$ form a basis of the lattice $(\mathbb{Z}^n)^*$.

Let us denote by ℓ_i the linear function $\lambda(v_i)$, for any $i \in [r]$. The characteristic map λ defines an m-dimensional family of hyperplane arrangements $\mathcal{AP}$ in the following way. For any $h = (h_1, \ldots, h_m) \in \mathbb{R}^m$, the arrangement $\mathcal{AP}(h)$ is given by

$$\mathcal{AP}(h) = \{H_1, \ldots, H_m\} \qquad \text{with} \quad H_i = \{\ell_i(x) = h_i\}.$$

We denote by X_h the union of all hyperplanes from $\mathcal{AP}(h)$.

Given a subset $I \in [m]$, we will denote by H_I the intersection

$$H_I = \bigcap_{j \in I} H_j.$$

It follows from the definition of the characteristic map that H_I is nonempty whenever the vertices v_j with $j \in I$ belong to the same simplex.

Let $\Delta^\perp$ be the *dual polyhedral complex* to Δ; we define a correspondence between the faces of $\Delta^\perp$ and the strata H_I in the following way: a face Γ_I of $\Delta^\perp$ dual to a simplex I of Δ is associated with the stratum H_I.

Definition 5.2. We say that a map $f \colon \Delta^\perp \to X_h$ is *subordinate* to a characteristic map λ if $f(\Gamma_I) \subset H_I$ for any face Γ_I of $\Delta^\perp$.

Theorem 5.3. *The space of maps $f \colon \Delta^\perp \to X_h$ subordinate to a characteristic map λ is a nonempty convex set. In particular, any two such maps are homotopic.*

Proof. First, let us show the second part of the statement, assuming that a map $f \colon \Delta^\perp \to X_h$ subordinate to the characteristic map λ exists. Observe that H_I is a convex set for any $I \subset [m]$. Therefore, for any two maps $f, f' \colon \Delta^\perp \to X_h$ subordinate to the characteristic map λ, any member of the linear homotopy between them is also subordinate to this characteristic map:

$$f_t := (1 - t)f + tf', \qquad t \in [0, 1].$$

Thus the space of maps $f \colon \Delta^\perp \to X_h$ compatible with the natural coverings of $\Delta^\perp$ and X_h is contractible (assuming it is nonempty).

To show the existence of such maps, we use the following construction. First, let us choose any inner product on $\mathbb{R}^n$. This defines a set of distinguished points $x_I \in H_I$ given by the orthogonal projection of the origin in $\mathbb{R}^n$ to the affine subspaces H_I. On the other hand, the points of the polyhedral complex $\Delta^\perp$ dual to the simplicial complex Δ, being the vertices of the barycentric subdivision Δ', are in a bijection with the simplices of Δ; hence they are labeled by subsets $I \subset [m]$.

We construct a map $f_h \colon \Delta^\perp \to X_h$ subordinate to the characteristic map λ as follows. First, we define the images of the above-mentioned points v_I of the complex $\Delta^\perp$ by the formula

$$f_h(v_I) = x_I,$$

and then we extend this map by linearity. Note that the map f_h just constructed is well defined, since (Δ, λ) is a characteristic pair (indeed, H_I is nonempty whenever I corresponds to a simplex in Δ) and is compatible with the covering of the complex $\Delta^\perp$ by the stars $\mathrm{St}(v_i)$ in Δ' of the vertices $v_i \in \Delta$, by construction. $\quad\square$

The family of maps $f_h \colon \Delta^\perp \to X_h$ has another nice property.

Corollary 5.4. *In the situation as before, one has $f_{h+h'} = f_h + f_{h'}$.*

Proof. The statement follows from the fact that the distinguished points x_I used in the above construction depend linearly on $h \in \mathbb{R}^n$:

$$x_{I, h+h'} = x_{I, h} + x_{I, h'}. \quad\square$$

With every affine hyperplane arrangement $\mathcal{AP}(h)$ we associate a chain $\Delta(h) = \sum_i W(U_i, f)U_i$, where U_i are the connected components of the complement $\mathbb{R}^n \setminus X_h$ and $f \colon \Delta^\perp \to X_h$ is any map subordinate to a characteristic map Λ. Since any two such maps are homotopic, the chain $\Delta(h)$ is well defined.

Definition 5.5. We will call the chain $\Delta(h)$ a *generalized virtual polytope* associated with the simplicial complex Δ, the characteristic map Λ, and the vector $h \in \mathbb{R}^m$. We denote by $\mathcal{P}_{\Delta, \Lambda} \simeq \mathbb{R}^m$ the space of all generalized virtual polytopes associated with the simplicial complex Δ and characteristic map Λ.

Remark 5.6. Classical virtual polytopes are piecewise linear functions defined not necessarily in the complements of unions of affine hyperplane arrangements (convex chains); hence they contain

more information than a chain $\Delta(h)$. However, in this paper we are interested only in the volumes of generalized virtual polytopes and integrals over them, so it is enough for us to work with the chain $\Delta(h)$. We will study other valuations on the space of generalized virtual polytopes in the future works.

Integration over generalized virtual polytopes. Let α be an $(n-1)$-form on $\mathbb{R}^n$ given by the formula

$$\alpha = P_1 \widehat{dx_1} \wedge \ldots \wedge dx_n + \ldots + P_n \, dx_1 \wedge \ldots \wedge \widehat{dx_n}.$$

Here, by $\widehat{dx_i}$ we mean that the term dx_i is missing. The following theorem is obvious.

Theorem 5.7. *If all the coefficients P_i of the form α are homogeneous polynomials of degree k (polynomials of degree at most k) on $\mathbb{R}^n$, then the function $\int_{\Delta^\perp} f^* \alpha$ is a homogeneous polynomial of degree $k + n - 1$ (a polynomial of degree at most $k + n - 1$) on the space of maps $f \colon \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ subordinate to the corresponding characteristic map.*

Proof. The proof is similar to that of Theorem 2.1, since by Corollary 5.4 the family of maps f_h can by chosen so that $f_{h_1} + f_{h_2} = f_{h_1 + h_2}$. $\square$

Let U be a bounded region in $\mathbb{R}^n \setminus \bigcup_{\mathcal{AP}(h)} H_i$ and $W(U, f)$ be the winding number for a map f as before. The next proposition follows from the Stokes theorem.

Proposition 5.8. *Let α be as before and $d\alpha = Q\omega$, where Q is a polynomial of degree at most k (homogeneous polynomial of degree k) on $\mathbb{R}^n$ and $\omega = dx_1 \wedge \ldots \wedge dx_n$ is the standard volume form on $\mathbb{R}^n$. Then the identity*

$$\sum W(U, f) \int_U Q\omega = \int_{\Delta^\perp} f^* \alpha$$

holds for the map $f \colon \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ subordinate to the corresponding characteristic map.

In particular, $\sum W(U, f) \int_U Q\omega$ is a polynomial of degree at most $k + n - 1$ (homogeneous polynomial of degree $k + n - 1$) on the space of maps $f \colon \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ subordinate to the corresponding characteristic map.

Given a polynomial Q on $\mathbb{R}^n$ and a generalized virtual polytope $f \colon \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$, let us denote by $I_Q(f)$ the integral

$$\sum W(U, f) \int_U Q\omega.$$

The following lemma computes the (mixed) partial derivatives of I_Q.

Lemma 5.9. *Let $f \colon \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ be a generalized virtual polytope associated with a simplicial complex Δ on s vertices. Suppose $I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, s\}$ is a subset such that the vertices $v_{i_1}, \ldots, v_{i_r}$ do not form a simplex in Δ, and let $k_1, \ldots, k_r$ be positive integers. Then we have*

$$\partial_{i_1}^{k_1} \ldots \partial_{i_r}^{k_r}(I_Q)(f) = 0.$$

However, if $r = n$ and the vertices $v_{i_1}, \ldots, v_{i_n}$ generate a simplex in Δ dual to a vertex $A \in \Delta^\perp$, then we have

$$\partial_I(I_Q)(f) = \operatorname{sgn}(I) Q(A) |\det(e_{i_1}, \ldots, e_{i_n})|.$$

Proof. By the linearity of derivation, it is enough to compute the partial derivatives for each summand $W(U, f) \int_U Q\omega$ separately.

In the first case, when the vertices $v_{i_1}, \ldots, v_{i_r}$ do not form a simplex in Δ, the intersection of the corresponding hyperplanes $H_{i_1}, \ldots, H_{i_r}$ does not correspond to a vertex of U, for any bounded region U in $\mathbb{R}^n \setminus \bigcup_{\mathcal{AP}(h)} H_i$ with $W(U, f) \neq 0$. Hence $\partial_{i_1}^{k_1} \ldots \partial_{i_r}^{k_r}(I_Q)(f) = 0$ by [11, Lemma 6.1].

On the other hand, if the vertices $v_{i_1}, \ldots, v_{i_n}$ generate a simplex in Δ, then there exists exactly one region U_i in $\mathbb{R}^n \setminus \bigcup_{\mathcal{AP}(h)} H_i$ that has the intersection $A = H_{i_1} \cap \ldots \cap H_{i_n}$ as its vertex. Then by [11, Lemma 6.1] we get

$$\partial_I(I_Q)(f) = \partial_I \int_{U_i} Q\omega = \operatorname{sgn}(I) Q(A) |\det(e_{i_1}, \ldots, e_{i_n})|. \qquad \square$$

As an immediate consequence of Lemma 5.9 we obtain the following statement.

Corollary 5.10. *Let* $f \colon \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ *be a generalized virtual polytope associated with a simplicial complex* Δ *on* s *vertices. Suppose* $I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, s\}$ *is a subset such that the vertices* $v_{i_1}, \ldots, v_{i_r}$ *do not form a simplex in* Δ, *and let* $k_1, \ldots, k_r$ *be positive integers. Then we have*

$$\partial_{i_1}^{k_1} \ldots \partial_{i_r}^{k_r} \operatorname{Vol}(f) = 0.$$

However, if $r = n$ *and the vertices* $v_{i_1}, \ldots, v_{i_n}$ *generate a simplex in* Δ *dual to a vertex* $A \in \Delta^\perp$, *then we have*

$$\partial_I \operatorname{Vol}(f)(f) = \operatorname{sgn}(I) |\det(e_{i_1}, \ldots, e_{i_n})|.$$

6. COHOMOLOGY OF GENERALIZED QUASITORIC MANIFOLDS

In this section we will describe the cohomology rings of a class of torus manifolds called generalized quasitoric manifolds. The results of this section were published in part in [10, 1]. Let $T \simeq (S^1)^n$ be a compact torus with character lattice M and $N = M^\vee$. Suppose K is an abstract simplicial complex of dimension $n-1$ on the vertex set $[m] = \{1, 2, \ldots, m\}$. Recall that its *moment–angle complex* $\mathcal{Z}_K$ is defined to be the $(m+n)$-dimensional cellular subspace in the unit polydisc $(D^2)^m \subset \mathbb{C}^m$ given by the formula $\bigcup_{I \in K} \prod_{i=1}^m Y_i$, where $Y_i = D^2$ if $i \in I$ and $Y_i = S^1$ otherwise.

There is a natural (coordinatewise) action of the compact torus $(S^1)^m$ on $\mathcal{Z}_K$, and the orbit space $\mathcal{Z}_K/(S^1)^m$ is homeomorphic to the cone over the barycentric subdivision of K.

In what follows we assume that $K = K_\Sigma$ is a *star-shaped sphere*, i.e., an intersection of a complete simplicial fan Σ in $\mathbb{R}^n \simeq N \otimes_{\mathbb{Z}} \mathbb{R}$ with the unit sphere $S^{n-1} \subset \mathbb{R}^n$. In this case, the moment–angle complex $\mathcal{Z}_K$ acquires a smooth structure (see [23]).

Let further $\Lambda \colon \Sigma(1) \to N$ be a *characteristic map*, i.e., a map such that the collection of vectors

$$\Lambda(\rho_1), \ldots, \Lambda(\rho_k)$$

can be completed to a basis of the cocharacter lattice N whenever $\rho_1, \ldots, \rho_k$ generate a cone in Σ. Then the $(m-n)$-dimensional subtorus $H_\Lambda := \ker \exp \Lambda \subset (S^1)^m$ acts freely on $\mathcal{Z}_K$ and the smooth manifold $X_{\Sigma, \Lambda} := \mathcal{Z}_K/H_\Lambda$ will be called a *generalized quasitoric manifold*.

Our description of the cohomology rings of $X_{\Sigma, \Lambda}$ will be given in three steps:

1. First, we give a cellular decomposition of $X_{\Sigma, \Lambda}$ of a special type and show that $H^*(X_{\Sigma, \Lambda})$ is generated by the classes dual to the classes of characteristic submanifolds of codimension 2 in $X_{\Sigma, \Lambda}$.

2. Then we deduce two sets of relations in the intersection ring of $X_{\Sigma, \Lambda}$ between the classes of characteristic submanifolds of codimension 2 in $X_{\Sigma, \Lambda}$.

3. Finally, we prove a topological version of the BKK theorem for $X_{\Sigma, \Lambda}$ and use it to get a Pukhlikov–Khovanskii type description for the integral cohomology ring $H^*(X_{\Sigma, \Lambda})$.

Remark 6.1. Note that steps 2 and 3 above could be successfully made in a much more general class of torus manifolds. However, in this more general case the algebra obtained by a Pukhlikov–Khovanskii description might be different from the cohomology ring. Indeed, the algebra

computed via the self-intersection polynomial is the Poincaré duality quotient of the subalgebra of the cohomology ring generated by the classes dual to the classes of characteristic submanifolds of codimension 2 (see [1] for details).

In what follows, we will always assume that our generalized quasitoric manifolds are *omnioriented*; as in the case of a quasitoric manifold, we say that $X_{\Sigma,\Lambda}$ is omnioriented if an orientation is specified for $X_{\Sigma,\Lambda}$ and for each of the m codimension 2 characteristic submanifolds D_i. The choice of this extra data is convenient for two reasons. First, it allows us to view the circle fixing D_i as an element in the lattice $N = \mathrm{Hom}(S^1, T^n) \simeq \mathbb{Z}^n$. But even more importantly, the choice of an omniorientation defines the fundamental class $[X_{\Sigma,\Lambda}]$ of $X_{\Sigma,\Lambda}$ as well as the cohomology classes $[D_i]$ dual to the characteristic submanifolds.

We further assume that $\Sigma \subset \mathbb{R}^n$ and $N_{\mathbb{R}} \simeq \mathbb{R}^n$ are endowed with orientation. This defines a sign for each collection of rays $\rho_{i_1}, \ldots, \rho_{i_n}$ forming a maximal cone of Σ in the following way. Let $I = \{i_1, \ldots, i_n\}$ be a set of indices ordered so that the collection of rays $\rho_{i_1}, \ldots, \rho_{i_n}$ is positively oriented in $\mathbb{R}^n$. Then

$$\mathrm{sgn}(I) = \det(\Lambda(\rho_{i_1}), \ldots, \Lambda(\rho_{i_n})) = \pm 1.$$

Finally, as before, with a characteristic pair (Σ, Λ) we associate a space of generalized virtual polytopes $\mathcal{P}_{\Sigma,\Lambda} \simeq \mathbb{R}^m$. With any generalized virtual polytope $\Delta(h) \in \mathcal{P}_{\Sigma,\Lambda}$ we associate an element of $H^2(X_{\Sigma,\Lambda})$ as follows:

$$\Delta(h) \mapsto h_1[D_1] + \ldots + h_m[D_m] \in H^2(X_{\Sigma,\Lambda}),$$

where $D_1, \ldots, D_m$ are the codimension 2 characteristic submanifolds oriented according to the given omniorientation of $X_{\Sigma,\Lambda}$.

6.1. Cellular decompositions of generalized quasitoric manifolds. To provide a cellular decomposition of the generalized quasitoric manifold $X_{\Sigma,\Lambda}$, let us first give a slightly different description of the moment–angle complex $\mathcal{Z}_K$ for a star-shaped sphere $K = K_{\Sigma}$. Observe that the moment–angle complex is given as a disjoint union of strata $\mathcal{Z}_K = \bigsqcup_{\sigma \in \Sigma} H_{\sigma}$, where

$$H_{\sigma} = \mathcal{Z}_K \cap \left(\bigcap_{\rho_i \in \sigma} \{z_i = 0\} \right) \cap \left(\bigcap_{\rho_j \notin \sigma} \{z_j \neq 0\} \right) \subset \mathbb{C}^m.$$

Our construction of a cell decomposition of $X_{\Sigma,\Lambda}$ is a slight generalization of the Morse-theoretic argument introduced in [14] and applied to quasitoric manifolds in [7]. Since we do not assume that Σ is a normal fan for a certain polytope, we cannot use the generic linear functions as in [14]. Instead, let us choose a vector $v \in \mathbb{R}^n$ in a general position with respect to Σ, i.e., a vector v which belongs to the interior of a full-dimensional cone of Σ.

Let $\tau_1, \ldots, \tau_s$ be cones of dimension n in Σ. For a maximal cone τ, we will say that a face σ of τ is *incoming* with respect to the vector v if the intersection $\tau \cap (\sigma + v)$ is unbounded. Let us further define the *index* $\mathrm{ind}(\tau)$ of a maximal cone τ to be the number of incoming rays of τ.

With each maximal cone τ we associate a disjoint union of open cells of $\mathcal{Z}_K$ via the formula

$$\widetilde{U}_{\tau} = \bigsqcup_{\sigma} H_{\sigma},$$

where the union is taken over all incoming faces σ of τ. Since each cone σ is incoming for a unique cone τ of maximal dimension, we get a cell decomposition

$$\mathcal{Z}_K = \bigsqcup_{i=1}^{s} \widetilde{U}_{\tau_i}.$$

It is easy to see that the cells $\widetilde{U}_\tau$ are invariant under the action of $H \simeq (S^1)^{m-n}$ and that

$$\widetilde{U}_\tau \simeq (D^2)^{\mathrm{ind}(\tau)} \times (S^1)^{m-n}.$$

Moreover, the action of H is free and transitive on the second factor in $(D^2)^{\mathrm{ind}(\tau)} \times (S^1)^{m-n}$; hence we get

$$X_{\Sigma,\Lambda} = \bigsqcup_{i=1}^{s} \widetilde{U}_{\tau_i}/H,$$

where $\widetilde{U}_{\tau_i}/H \simeq (D^2)^{\mathrm{ind}(\tau_i)}$

Theorem 6.2. *Let $X_{\Sigma,\Lambda}$ be a generalized quasitoric manifold. Then $X_{\Sigma,\Lambda}$ has a cellular decomposition with only even-dimensional cells. The cells in this decomposition are in a bijection with the maximal cones τ in Σ. The dimension of the cell corresponding to a cone τ is $2\,\mathrm{ind}(\tau)$.*

Corollary 6.3. *The Euler characteristic of the manifold $X_{\Sigma,\Lambda}$ is equal to the number of maximal cones in Σ.*

6.2. Relations between characteristic submanifolds. In this subsection we will deduce two types of relations between classes of codimension 2 characteristic submanifolds in the intersection ring of a manifold $X_{\Sigma,\Lambda}$. In the following proposition we show that the Stanley–Reisner relations hold in $H^*(X_{\Sigma,\Lambda})$.

Proposition 6.4. *For codimension 2 characteristic submanifolds $D_{i_1}, \ldots, D_{i_n}$, in the cohomology ring of a generalized quasitoric manifold $X_{\Sigma,\Lambda}$ one has*

$$[D_{i_1}] \ldots [D_{i_n}] = \begin{cases} \mathrm{sgn}(I)[X_{\Sigma,\Lambda}]^* & \text{if } \rho_{i_1}, \ldots, \rho_{i_n} \text{ form a cone in } \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Indeed, in the cohomology ring of the generalized quasitoric manifold $X_{\Sigma,\Lambda}$, we have $[D_{i_1}] \ldots [D_{i_n}] = (-1)^v [X_{\Sigma,\Lambda}]^*$, where $v = D_{i_1} \cap \ldots \cap D_{i_n} \in X_{\Sigma,\Lambda}$ is a fixed point and $(-1)^v$ is its sign obtained by comparing two orientations on $\mathcal{T}_v X_{\Sigma,\Lambda}$: one induced by the coorientations of the characteristic submanifolds D_i and the other induced by the representation of $T^n := T^m/H$ in the tangent space $\mathcal{T}_v X_{\Sigma,\Lambda} \cong \mathbb{C}^n$.

On the other hand, the weights of the tangential representation of the compact torus T^n at the fixed point v form a lattice basis dual to the basis $(\Lambda(\rho_{i_1}), \ldots, \Lambda(\rho_{i_n}))$. Therefore, $(-1)^v = \det(\Lambda(\rho_{i_1}), \ldots, \Lambda(\rho_{i_n})) = \mathrm{sgn}(I)$, which completes the proof. $\square$

To obtain linear relations, we need to analyze further the construction of generalized quasitoric manifolds. There are natural $(S^1)^m$-equivariant line bundles $L_1, \ldots, L_m$ on $\mathcal{Z}_K$. For each integer vector $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m$, the tensor product

$$L_{\mathbf{k}} = L_1^{k_1} \otimes \ldots \otimes L_m^{k_m}$$

descends to a complex line bundle $\widetilde{L}_{\mathbf{k}}$ on $X_{\Sigma,\Lambda}$. Moreover, if $\mathbf{k} \in \mathbb{Z}^m$ is such that the corresponding character acts trivially on $H_\Lambda \subset (S^1)^m$, the descendant bundle $\widetilde{L}_{\mathbf{k}}$ is topologically trivial.

It is easy to see that there is a smooth section of $\widetilde{L}_{\mathbf{k}}$ with the degenerate locus given by $\sum_{i=1}^{m} k_i [D_i]$. By exactness of the sequence

$$0 \to M \xrightarrow{\Lambda^*} \mathbb{Z}^m \to M_{H_\Lambda} \to 0,$$

the characters $\mathbf{k}$ acting trivially on H_Λ are identified with the character lattice M of T with $k_i = \chi(v_i)$ for $\chi \in M$ and $v_i = \Lambda(\rho_i)$. Thus we obtain the following proposition.

Proposition 6.5. *For any character $\chi \in M$, the following linear relation in $H^2(X_{\Sigma,\Lambda})$ holds:*

$$\sum_{i=1}^{m} \chi(v_i)[D_i] = 0,$$

where $v_i := \Lambda(\rho_i)$ for $1 \le i \le m$.

Proof. Indeed, the descendant complex line bundle $\widetilde{L}_{\chi(v_1),\ldots,\chi(v_m)}$ is trivial, and hence its first Chern class is equal to zero:

$$c_1(\widetilde{L}_{\chi(v_1),\ldots,\chi(v_m)}) = \sum_{i=1}^{m} \chi(v_i)[D_i] = 0. \qquad \square$$

6.3. Topological version of the BKK theorem. Let us start with an important observation: to describe the cohomology ring of a generalized quasitoric manifold, it is enough to compute the *self-intersection polynomial*

$$h_1[D_1] + \ldots + h_m[D_m] \mapsto \left\langle (h_1[D_1] + \ldots + h_m[D_m])^m, [X_{\Sigma,\Lambda}] \right\rangle$$

on the space of all linear combinations of classes of codimension 2 characteristic submanifolds. This is the subject of the following theorem. Theorem 6.6 is closely related to [10, Lemma 8.6].

Theorem 6.6. *Let $X_{\Sigma,\Lambda}$ be a generalized quasitoric manifold with codimension 2 characteristic submanifolds $D_1,\ldots,D_m$. Then the following identity holds:*

$$\left\langle (h_1[D_1] + \ldots + h_m[D_m])^m, [X_{\Sigma,\Lambda}] \right\rangle = n!\, \mathrm{Vol}(f_h),$$

where $f_h \in \mathcal{P}_{\Sigma,\Lambda}$ is a generalized virtual polytope associated with the simplicial complex K_Σ, the characteristic map Λ, and the set of parameters $h = (h_1,\ldots,h_m)$.

Proof. Let us identify the space of all linear combinations $h_1[D_1] + \ldots + h_m[D_m]$ with the space of generalized virtual polytopes $\mathcal{P}_{\Sigma,\Lambda}$. Under this identification, both self-intersection and volume functions are homogeneous polynomials of degree n on $\mathcal{P}_{\Sigma,\Lambda}$. Let us denote them by $S \colon \mathcal{P}_{\Sigma,\Lambda} \to \mathbb{R}$ and $\mathrm{Vol} \colon \mathcal{P}_{\Sigma,\Lambda} \to \mathbb{R}$, respectively.

To show the equality $S(h) = n!\,\mathrm{Vol}(h)$, it is enough to prove the equality of all (mixed) partial derivatives of S and Vol of degree n:

$$\partial_{i_1}^{k_1} \ldots \partial_{i_s}^{k_s} S(h) = n!\, \partial_{i_1}^{k_1} \ldots \partial_{i_s}^{k_s} \mathrm{Vol}(h),$$

where $\partial_{i_j} = \partial/\partial h_{i_j}$ and $\sum_{j=1}^{s} k_{i_j} = n$.

Let us call the number $\sum_{i=1}^{s}(k_i - 1)$ the *multiplicity of the monomial* $\partial_{i_1}^{k_1} \ldots \partial_{i_s}^{k_s}$. In particular, a monomial has multiplicity 0 if and only if it is square free. We will prove the equality of mixed partial derivatives by induction on the multiplicity of a differential monomial.

For square free monomials, the equality follows from the first part of Corollary 5.10 and Proposition 6.4. Indeed, by Corollary 5.10, in the case when $r = n$ and the vertices $v_{i_1},\ldots,v_{i_n}$ form a simplex in Δ dual to a vertex $A \in \Delta^\perp$, we have

$$\partial_{i_1} \ldots \partial_{i_n} \mathrm{Vol}(h) = \begin{cases} \mathrm{sgn}(i_1,\ldots,i_n) & \text{if } \rho_{i_1},\ldots,\rho_{i_n} \text{ span a cone in } \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, $\partial_{i_1} \ldots \partial_{i_n} S(h)$ is equal to the coefficient of $t_{i_1} \ldots t_{i_n}$ in the polynomial $S(h + (t_1,\ldots,t_m))$. We get

$$S(h + (t_1,\ldots,t_n)) = \left\langle ((h_1 + t_1)[D_1] + \ldots + (h_m + t_m)[D_m])^m, [X_{\Sigma,\Lambda}] \right\rangle$$

$$= t_{i_1} \ldots t_{i_n} \cdot n! \cdot \left\langle [D_{i_1}] \ldots [D_{i_n}], [X_{\Sigma,\Lambda}] \right\rangle + \ldots.$$

Hence by Proposition 6.4 we get

$$\partial_{i_1}\dots\partial_{i_n}S(h) = \begin{cases} n!\operatorname{sgn}(i_1,\dots,i_n) & \text{if } \rho_{i_1},\dots,\rho_{i_n} \text{ span a cone in } \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let us assume that the equality of mixed partial derivatives holds for all differential monomials of multiplicity $r-1$. Let $\partial_{i_1}^{k_1}\dots\partial_{i_s}^{k_s}$ be a differential monomial of multiplicity r with $k_1 \geq 1$. We can assume that $\rho_{i_1},\dots,\rho_{i_s}$ span a cone in Σ, since otherwise

$$\partial_{i_1}^{k_1}\dots\partial_{i_s}^{k_s}S(h) = n!\,\partial_{i_1}^{k_1}\dots\partial_{i_s}^{k_s}\operatorname{Vol}(h) = 0.$$

In that case, there exists a character $\chi \in M$ such that

$$\langle\chi,\Lambda(\rho_{i_1})\rangle = 1, \quad \langle\chi,\Lambda(\rho_{i_2})\rangle = 0, \quad \dots, \quad \langle\chi,\Lambda(\rho_{i_s})\rangle = 0.$$

Therefore, since the volume is invariant under the translation of a generalized virtual polytope, we get

$$\partial_{i_1}^{k_1}\dots\partial_{i_s}^{k_s}\operatorname{Vol}(h) = -\sum_{l\neq i_j}\langle\chi,\Lambda(\rho_l)\rangle\,\partial_l\partial_{i_1}^{k_1-1}\dots\partial_{i_s}^{k_s}\operatorname{Vol}(h)$$

and similarly by Proposition 6.5

$$\partial_{i_1}^{k_1}\dots\partial_{i_s}^{k_s}S(h) = -\sum_{l\neq i_j}\langle\chi,\Lambda(\rho_l)\rangle\,\partial_l\partial_{i_1}^{k_1-1}\dots\partial_{i_s}^{k_s}S(h).$$

Moreover, the differential monomials on the right-hand side of the expressions above have multiplicities less than r, so the equality

$$\partial_{i_1}^{k_1}\dots\partial_{i_s}^{k_s}S(h) = n!\,\partial_{i_1}^{k_1}\dots\partial_{i_s}^{k_s}\operatorname{Vol}(h)$$

follows from the induction hypothesis. $\square$

We will finish this subsection by providing a different interpretation of Theorem 6.6. Let us first recall the classical interpretation of the BKK theorem for toric varieties. The Newton polyhedron $\Delta(f) \subset \mathbb{R}^n$ of a Laurent polynomial $f = \sum a_i x^{k_i}$ is the convex hull of the vectors k_i with $a_i \neq 0$. For a fixed polytope Δ, let E_Δ be a finite-dimensional vector space of Laurent polynomials f such that $\Delta(f) \subset \Delta$. The BKK theorem gives the number of solutions of the system $f_1 = \dots = f_n = 0$ in $(\mathbb{C}^*)^n$ for general Laurent polynomials with fixed Newton polyhedra $\Delta_1,\dots,\Delta_n$.

Theorem 6.7 (BKK theorem). *Let $\Delta_1,\dots,\Delta_n$ — be fixed integer polyhedra and $f_1,\dots,f_n$ be generic Laurent polynomials such that $\Delta(f_i) \subset \Delta_i$ for $1 \leq i \leq n$. Then all the solutions of the system $f_1 = \dots = f_n = 0$ in $(\mathbb{C}^*)^n$ are nondegenerate and the number of solutions is equal to*

$$n!\operatorname{Vol}(\Delta_1,\dots,\Delta_n),$$

where Vol *is the mixed volume function of virtual polytopes.*

One can reformulate Theorem 6.6 in a similar way. Let $\Delta_1,\dots,\Delta_n$ be generalized virtual polytopes in $\mathcal{P}_{\Sigma,\Lambda}$ associated with a generalized quasitoric manifold $X_{\Sigma,\Lambda}$. Let L_{Δ_i} be a line bundle associated with the generalized virtual polytope Δ_i and let $E_{\Delta_i} = \Gamma(X_{\Sigma,\Lambda}, L_{\Delta_i})$ be the space of smooth sections of L_{Δ_i}. Then Theorem 6.6 can be reformulated in the following way.

Theorem 6.8. *Let $\Delta_1,\dots,\Delta_n$ be fixed generalized virtual polytopes from $\mathcal{P}_{\Sigma,\Lambda}$ and $s_1,\dots,s_n$ be generic sections of $L_{\Delta_1},\dots,L_{\Delta_n}$ with $s_i \in E_{\Delta_i}$ for $1 \leq i \leq n$. Then all the solutions of the system $s_1 = \dots = s_n = 0$ in $X_{\Sigma,\Lambda}$ are nondegenerate and the number of solutions counted with signs is equal to*

$$n!\operatorname{Vol}(\Delta_1,\dots,\Delta_n),$$

where Vol *is the mixed volume function of generalized virtual polytopes.*

Remark 6.9. Note that in the algebraic case the multiplicity of each nondegenerate root is equal to 1; however, in the case of smooth sections $s_i \in \Gamma(X_{\Sigma,\Lambda}, L_{\Delta_i})$, the multiplicity of a nondegenerate root might be equal to -1. Nevertheless, the number of solutions counted with signs can still be computed in terms of a mixed volume.

6.4. Pukhlikov–Khovanskii type description. In this subsection we use the approach introduced by A. V. Pukhlikov and the third author for the computation of cohomology rings. The key ingredient of such a description is an exact computation of Macaulay inverse systems for graded algebras with Poincaré duality generated in degree 1.

We will call a graded commutative algebra $A = \bigoplus_{i=0}^{n} A_i$ over a field $\mathbb{K}$ of characteristic 0 a *Poincaré duality algebra* if

- $A_0 \simeq A_n \simeq \mathbb{K}$;
- the bilinear map $A_i \times A_{n-i} \to A_n$ is nondegenerate for any $i = 0, \dots, n$ (Poincaré duality).

The main example of a Poincaré duality algebra arises as follows. Let X be a smooth closed orientable manifold of dimension $2n$. Then the algebra of even-degree cohomology classes $A = \bigoplus_{i=0}^{n} H^{2i}(X)$ is a Poincaré duality algebra. In particular, since for a generalized quasitoric manifold $X_{\Sigma,\Lambda}$ one has $H^{2i+1}(X_{\Sigma,\Lambda}) = 0$ for all $i \geq 0$, its cohomology ring $H^*(X_{\Sigma,\Lambda})$ is also a Poincaré duality algebra. The next theorem yields a description of Poincaré duality algebras.

Theorem 6.10. *Let A be a Poincaré duality algebra generated (as an algebra) by the elements from $A_1 = \mathbb{K}\langle v_1, \dots, v_r \rangle$ (i.e., by elements of degree 1). Then*

$$A \simeq \mathbb{K}[t_1, \dots, t_r] / \left\{ p(t_1, \dots, t_r) \in \mathbb{K}[t_1, \dots, t_r] \colon p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}\right) f(x_1, \dots, x_r) = 0 \right\},$$

where we identify A_1 with $\mathbb{K}^r$ via a basis $v_1, \dots, v_r$ and $f \colon A_1 \simeq \mathbb{K}^r \to \mathbb{K}$ is a polynomial given by the formula

$$f(x_1, \dots, x_r) = (x_1 v_1 + \dots + x_r v_r)^n \in A_n \simeq \mathbb{K}.$$

Theorem 6.10 was used in [25] to give a description of the cohomology ring of a smooth projective toric variety. Later, it was used in [13] to provide a description of the cohomology rings of full flag varieties G/B. A more general version of Theorem 6.10 has been obtained recently in [16] and used in [11, 15] to give a description of the cohomology rings of toric and quasitoric bundles.

Theorem 6.10 admits a coordinate-free reformulation. Indeed, the ring $\mathbb{K}[t_1, \dots, t_r]$ in Theorem 6.10 can be identified with the ring of differential operators with constant coefficients $\mathrm{Diff}(A_1)$ on A_1. Hence the description of the algebra A becomes

$$A \simeq \mathrm{Diff}(A_1)/\mathrm{Ann}(f),$$

where $\mathrm{Ann}(f) = \{ D \in \mathrm{Diff}(A_1) \colon D \cdot f = 0 \}$ is the annihilator ideal of f.

Theorem 6.11. *Let $X_{\Sigma,\Lambda}$ be a generalized quasitoric manifold and let $\mathcal{P}_{\Sigma,\Lambda}$ be the space of generalized virtual polytopes associated with it. Then the cohomology ring $H^*(X_{\Sigma,\Lambda})$ can be computed as*

$$H^*(X_{\Sigma,\Lambda}) = \mathrm{Diff}(\mathcal{P}_{\Sigma,\Lambda})/\mathrm{Ann}(\mathrm{Vol}),$$

where $\mathrm{Diff}(\mathcal{P}_{\Sigma,\Lambda})$ is the ring of differential operators with constant coefficients on $\mathcal{P}_{\Sigma,\Lambda}$ and $\mathrm{Ann}(\mathrm{Vol})$ is the annihilator ideal of the volume polynomial.

Proof. By Theorem 6.2, the cohomology ring $H^*(X_{\Sigma,\Lambda})$ is generated by the classes of codimension 2 characteristic submanifolds in $X_{\Sigma,\Lambda}$. Hence there exists a surjection $\mathrm{Diff}(\mathcal{P}_{\Sigma,\Lambda}) \to H^*(X_{\Sigma,\Lambda})$ with a kernel given, by Theorem 6.10, as the annihilator ideal of the self-intersection polynomial $S(h)$

of classes of codimension 2 characteristic submanifolds. However, by Theorem 6.6, $S(h) = n! \operatorname{Vol}(h)$ and hence

$$H^*(X_{\Sigma,\Lambda}) = \operatorname{Diff}(\mathcal{P}_{\Sigma,\Lambda})/\operatorname{Ann}(S) = \operatorname{Diff}(\mathcal{P}_{\Sigma,\Lambda})/\operatorname{Ann}(\operatorname{Vol}). \quad \square$$

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